

# Liquidity Aggregation Across Markets

## Effective Depth, Cross-Margining, and Corridor-Scaling Bounds

Raez Lorgat

October 27, 2025

### Abstract

Liquidity aggregation is used to describe two distinct phenomena. The first is *within-venue aggregation*: multiple markets clear against a shared collateral pool and shared router, so the same risk capital supports more executable notional than in fragmented isolated books. The second is *corridor-network expansion*: additional sovereign zones are connected by negotiated corridors. They are not the same object and they obey different scaling laws.

For within-venue aggregation we study a quadratic portfolio-risk model with margin functional  $M(q) = \sqrt{q^\top \Sigma q}$  on a position vector  $q \in \mathbb{R}^m$  and positive-semidefinite risk matrix  $\Sigma$ . The effective depth available to market  $i$  under free margin  $F$  is

$$L_i^{\text{eff}}(q, F, \Sigma) = \frac{-b_i(q) + \sqrt{b_i(q)^2 + \sigma_i^2(2R(q)F + F^2)}}{\sigma_i^2},$$

where  $R(q) = \sqrt{q^\top \Sigma q}$ ,  $\sigma_i^2 = \Sigma_{ii}$ , and  $b_i(q) = e_i^\top \Sigma q$ . Super-additive depth is present exactly when the new trade is not  $\Sigma$ -metric-aligned with the existing book,  $b_i(q) < \sigma_i R(q)$ ; equivalently, by Cauchy-Schwarz, whenever  $e_i$  is not a non-negative multiple of  $q$  in the  $\Sigma$ -metric. In particular, any incoming trade that offsets the book ( $b_i(q) < 0$ ) yields a strict bonus, and the bonus grows in  $|b_i(q)|$ . In the symmetric equicorrelated case with isolated per-market depth  $V$  and average correlation  $\rho \in [0, 1]$ , total executable notional across  $n$  markets is

$$L(n, \rho) = nV \sqrt{\frac{n}{1 + (n-1)\rho}},$$

which is strictly super-additive for  $\rho < 1$ , additive at  $\rho = 1$ , and bounded above by  $n^{3/2}V$ . This replaces unsupported quadratic ansätze such as  $nV + \alpha n(n-1)$ .

We then analyze three pressure points. First, cross-margining across spot, perpetual, and event-linked positions produces real offsets only when credits are computed from a single positive-semidefinite portfolio-risk matrix; pairwise rebate schedules that are not equivalent to a single matrix can double-count. Second, aggregation can dilute fees per unit displayed depth unless routed volume rises sufficiently fast; LP gains depend on fee growth, fixed-cost amortization, and incremental adverse selection or loss-versus-rebalancing cost. Third, concentrated withdrawals and fragmentation admit explicit first-order damage bounds.

Finally, corridor networks are treated separately. Following Odlyzko and Tilly's critique of Metcalfe scaling, corridor value is weakly superlinear and not quadratic. Under order-of-magnitude negotiation-cost assumptions consistent with bilateral investment treaty and preferential trade agreement formation data [7, 8], realizable corridor growth is governed by sparse hub-and-spoke construction and cost-constrained  $O(n \log n)$  scaling, not by  $n^2$  pair counting.

**Contents**

<b>1</b>	<b>Introduction</b>	<b>3</b>
<b>2</b>	<b>Model Primitives</b>	<b>4</b>
2.1	Venue, positions, and margin . . . . .	4
2.2	Isolated baseline and depth . . . . .	4
<b>3</b>	<b>Within-Venue Effective Depth</b>	<b>4</b>
3.1	Single-market effective depth against an existing book . . . . .	4
3.2	Symmetric equicorrelation model . . . . .	6
<b>4</b>	<b>Cross-Margining Across Instrument Types</b>	<b>7</b>
4.1	Offsets are real only under one risk matrix . . . . .	7
4.2	Stress correlation and the Li-copula caveat . . . . .	8
<b>5</b>	<b>LP Incentives and the Fragmentation Trade-Off</b>	<b>9</b>
5.1	Fees per unit displayed depth . . . . .	9
5.2	Profitability on risk capital . . . . .	9
<b>6</b>	<b>Adversarial Stress Bounds</b>	<b>10</b>
6.1	Concentrated withdrawals . . . . .	10
6.2	Fragmentation after stress . . . . .	10
<b>7</b>	<b>Corridor Networks Are Different</b>	<b>11</b>
<b>8</b>	<b>Limitations</b>	<b>12</b>
<b>9</b>	<b>Conclusion</b>	<b>13</b>

## 1 Introduction

The phrase *liquidity aggregation* is applied to distinct phenomena. One paper will use it to mean that many markets clear against one collateral pool. Another will use it to mean that many jurisdictions are connected by bilateral corridors. Those are different mechanisms, driven by different constraints, and they should not share a scaling claim.

This paper studies the two objects separately.

- (i) **Within-venue aggregation.** A venue clears many markets against one portfolio-margin engine and one router. The relevant question is: how much executable notional can a fixed amount of risk capital support when positions offset across markets?
- (ii) **Corridor-network expansion.** A system links multiple sovereign zones through negotiated corridors. The relevant question is: how many corridors are economically and politically feasible, and how does realizable value scale with the number of zones?

Cross-chain or cross-zone transport belongs to the second category even when execution inside each zone is internally aggregated. The within-venue theorem of this paper is not a theorem about bridge topology.

The first object is a microstructure and clearing problem. The second is a network-formation problem with large fixed costs and long gestation. Conflating them produces two recurring errors. The first error is to present a stylized depth bonus such as  $nV + \alpha n(n-1)$  as if it were a theorem. The second error is to transfer Metcalfe-style  $n^2$  language from abstract communication networks to corridor systems whose edges require sovereign negotiation.

The paper makes five claims.

- (C1) Under quadratic portfolio risk, effective depth in market  $i$  admits a closed form and is super-additive exactly when the incoming trade is not Cauchy-Schwarz-saturated against the current book:  $b_i(q) < \sigma_i R(q)$ . A sufficient strong-bonus condition is  $b_i(q) < 0$  (incoming trade directly offsets the book).
- (C2) In the symmetric equicorrelation model, total executable notional across  $n$  markets is  $L(n, \rho) = nV \sqrt{n / (1 + (n-1)\rho)}$ . This is a theorem under stated assumptions. It is linear-to- $n^{3/2}$  in  $n$ , never quadratic.
- (C3) Cross-margin offsets are real only when computed from a single generator: one positive-semidefinite risk matrix in the quadratic model, or one shared scenario family in scenario-margin engines. Rebate schedules that do not factor through a single generator can double-count the same hedge.
- (C4) Aggregation is not automatically LP-positive. It can dilute fee flow per unit displayed depth unless routed volume, fixed-cost amortization, or both dominate the extra risk cost.
- (C5) Corridor scaling is a separate sparse-network problem. Under cost-constrained corridor formation the appropriate benchmark is weakly superlinear  $O(n \log n)$  growth, following Odlyzko and Tilly [6], not Metcalfe  $n^2$ .

The paper sits at the intersection of three literatures. Optimal routing across constant-function market makers is developed by Angeris, Evans, Chitra, and Boyd [1]. Multi-lateral netting and clearing efficiency are analyzed by Duffie and Zhu [2]. Liquidity-provider loss versus rebalancing is formalized by Milionis, Moallemi, Roughgarden, and Zhang [5]. The tail-risk caution for copula and correlation-based aggregation follows Embrechts, McNeil, and Straumann [3]; the Gaussian-copula register associated with Li [4] is a dependence model, not a theorem that tail dependence is negligible.

## 2 Model Primitives

### 2.1 Venue, positions, and margin

**Definition 2.1** (Unified position space). Let  $\mathcal{P} = \{1, \dots, m\}$  index cleared positions across all markets and instrument types on a single venue. Coordinates can represent spot, perpetual, event-linked, or other claims. A participant's marked-to-risk position is a vector  $q = (q_1, \dots, q_m) \in \mathbb{R}^m$ .

**Definition 2.2** (Quadratic portfolio-risk functional). Let  $\Sigma \in \mathbb{R}^{m \times m}$  be positive semidefinite with diagonal entries  $\sigma_i^2 = \Sigma_{ii} > 0$ . The portfolio-risk functional is

$$M(q) = \sqrt{q^\top \Sigma q}. \quad (1)$$

For a free-margin budget  $F > 0$ , the feasible post-trade set is

$$\mathcal{S}(q, F, \Sigma) = \{x \in \mathbb{R}_{\geq 0}^m : M(q + x) \leq M(q) + F\}.$$

**Remark 2.3** (Scope of the risk model). Equation (1) is exact for Gaussian or more general elliptical central-body models and is a local approximation to many scenario-margin engines. The results derived from it are statements *under that model class*. When tail dependence exceeds what  $\Sigma$  captures, the matrix must be replaced by a stressed matrix or by a copula-derived upper bound; Section 4 makes that caveat explicit.

### 2.2 Isolated baseline and depth

**Definition 2.4** (Isolated depth). For market coordinate  $i$ , the isolated depth available under free margin  $F$  is

$$V_i(F) := \frac{F}{\sigma_i}. \quad (2)$$

This is the executable size when the book is empty and the new trade is margined in isolation.

**Remark 2.5** (Displayed depth versus effective depth). The paper studies *effective depth*: how much additional notional the margin engine can support. Displayed order-book depth and AMM reserve depth remain relevant for execution, but they are not the object of the super-additivity theorem below. Within-venue aggregation is a balance-sheet and routing statement, not a claim that one market's local order book automatically contains all liquidity everywhere else.

## 3 Within-Venue Effective Depth

### 3.1 Single-market effective depth against an existing book

**Definition 3.1** (Effective depth). Fix market coordinate  $i \in \mathcal{P}$  and let  $e_i$  denote the  $i$ th standard basis vector. The *effective depth* available to market  $i$  against current book  $q$  and free margin  $F$  is

$$L_i^{\text{eff}}(q, F, \Sigma) := \sup\{x \geq 0 : M(q + xe_i) \leq M(q) + F\}. \quad (3)$$

**Proposition 3.2** (Closed form for effective depth). *Let*

$$R(q) := \sqrt{q^\top \Sigma q}, \quad b_i(q) := e_i^\top \Sigma q.$$

*Then*

$$L_i^{\text{eff}}(q, F, \Sigma) = \frac{-b_i(q) + \sqrt{b_i(q)^2 + \sigma_i^2(2R(q)F + F^2)}}{\sigma_i^2}. \quad (4)$$

*Proof.* By Definition 3.1,  $x = L_i^{\text{eff}}(q, F, \Sigma)$  is the largest non-negative solution of

$$\sqrt{(q + xe_i)^\top \Sigma (q + xe_i)} \leq R(q) + F.$$

Squaring both sides gives

$$q^\top \Sigma q + 2xe_i^\top \Sigma q + x^2 e_i^\top \Sigma e_i \leq q^\top \Sigma q + 2R(q)F + F^2.$$

Using  $e_i^\top \Sigma q = b_i(q)$  and  $e_i^\top \Sigma e_i = \sigma_i^2$ , the feasible  $x$  satisfy

$$\sigma_i^2 x^2 + 2b_i(q)x - (2R(q)F + F^2) \leq 0.$$

The positive root of this quadratic is exactly (4).  $\square$

**Corollary 3.3** (Super-additive depth condition and upper bound). *Let  $V_i(F) = F/\sigma_i$  be the isolated depth from Definition 2.4. Then:*

- (a)  $L_i^{\text{eff}}(0, F, \Sigma) = V_i(F)$ .
- (b) For  $q \neq 0$ ,  $L_i^{\text{eff}}(q, F, \Sigma) \geq V_i(F)$  with equality if and only if  $b_i(q) = \sigma_i R(q)$  (the Cauchy-Schwarz upper-saturation case in which  $e_i$  is a non-negative multiple of  $q$  in the  $\Sigma$ -metric). Equivalently, super-additivity  $L_i^{\text{eff}}(q, F, \Sigma) > V_i(F)$  holds iff  $b_i(q) < \sigma_i R(q)$ .
- (c) The diversification bonus is monotone decreasing in  $b_i(q)$ ; in particular, whenever the incoming trade offsets the existing book in the portfolio metric ( $b_i(q) < 0$ ), the bonus is strict and grows in  $|b_i(q)|$ .
- (d)  $L_i^{\text{eff}}(q, F, \Sigma) \leq (R(q) + F)/\sigma_i$ , with equality exactly in the lower-saturation case  $b_i(q) = -\sigma_i R(q)$  (direct-offset limit).

*Proof.* Part (a) is immediate from (4) with  $q = 0$ .

For part (b), the inequality  $L_i^{\text{eff}}(q, F, \Sigma) > F/\sigma_i$  is

$$-b_i(q) + \sqrt{b_i(q)^2 + \sigma_i^2(2R(q)F + F^2)} > \sigma_i F,$$

equivalently

$$\sqrt{b_i(q)^2 + \sigma_i^2(2R(q)F + F^2)} > \sigma_i F + b_i(q).$$

Two cases.

*Case 1:*  $\sigma_i F + b_i(q) < 0$ . The right-hand side is negative and the left-hand side is non-negative, so the strict inequality holds automatically. In this case  $b_i(q) < -\sigma_i F < 0 \leq \sigma_i R(q)$ , so  $b_i(q) < \sigma_i R(q)$  also holds.

*Case 2:*  $\sigma_i F + b_i(q) \geq 0$ . Both sides are non-negative, so squaring preserves the inequality. After squaring and cancelling  $\sigma_i^2 F^2 + b_i(q)^2$ ,

$$2\sigma_i^2 R(q)F > 2\sigma_i F b_i(q),$$

which (using  $\sigma_i > 0$  and  $F > 0$ ) simplifies to  $\sigma_i R(q) > b_i(q)$ .

Combining both cases,  $L_i^{\text{eff}}(q, F, \Sigma) > V_i(F)$  holds exactly when  $b_i(q) < \sigma_i R(q)$ . By the Cauchy-Schwarz bound  $|b_i(q)| \leq \sigma_i R(q)$ , the strict inequality fails only at the upper-saturation point  $b_i(q) = \sigma_i R(q)$ ; at that point a direct substitution in (4) gives  $L_i^{\text{eff}} = (-\sigma_i R(q) + \sigma_i(R(q) + F))/\sigma_i^2 = F/\sigma_i = V_i(F)$ . This proves part (b).

For part (c), differentiate (4) with respect to  $b_i$  (treating  $R$  and  $\sigma_i$  as constants):

$$\frac{\partial L_i^{\text{eff}}}{\partial b_i} = \frac{-1 + b_i/\sqrt{b_i^2 + \sigma_i^2(2RF + F^2)}}{\sigma_i^2} \leq \frac{-1 + |b_i|/\sqrt{b_i^2}}{\sigma_i^2} = 0,$$

with strict inequality whenever  $2R(q)F + F^2 > 0$ , i.e. for any  $R(q) > 0$  or  $F > 0$ . Hence  $L_i^{\text{eff}}$  is strictly decreasing in  $b_i$ . In the directly offsetting regime  $b_i(q) < 0$ , monotonicity gives  $L_i^{\text{eff}} > L_i^{\text{eff}}|_{b_i=0} > L_i^{\text{eff}}|_{b_i=\sigma_i R} = V_i(F)$ .

For part (d), Cauchy-Schwarz gives  $|b_i(q)| \leq \sigma_i R(q)$ , hence

$$L_i^{\text{eff}}(q, F, \Sigma) \leq \frac{\sigma_i R(q) + \sqrt{\sigma_i^2 R(q)^2 + \sigma_i^2 (2R(q)F + F^2)}}{\sigma_i^2} = \frac{R(q) + F}{\sigma_i},$$

with equality exactly at  $b_i(q) = -\sigma_i R(q)$ .  $\square$

**Remark 3.4** (Scope of super-additivity). The sharp super-additivity condition is geometric:  $b_i(q) < \sigma_i R(q)$ . By Cauchy-Schwarz, the only configuration that produces no bonus is the upper-saturation case  $b_i(q) = \sigma_i R(q)$ , where the incoming direction  $e_i$  is a non-negative  $\Sigma$ -metric multiple of the existing book  $q$ . Generic incoming trades - including those that are  $\Sigma$ -orthogonal to the book ( $b_i(q) = 0$ ) - generate a strict diversification bonus over isolated margin. Direct offsets give a sufficient condition, not the whole phenomenon. The cleanest economic sufficient condition remains  $b_i(q) < 0$ , where the new trade cancels variance in the existing book and the bonus is strictly larger than at  $b_i(q) = 0$ . The claim is local and balance-sheet based. It does not say that every pairwise market becomes as deep as the venue-wide aggregate. It does not say that order-book or AMM reserves can be counted multiple times.

### 3.2 Symmetric equicorrelation model

The closed form above is exact but state dependent. To understand scaling in  $n$  we pass to a symmetric model.

**Definition 3.5** (Symmetric equicorrelation regime). Assume:

- (E1)  $n$  markets have identical marginal volatility  $\sigma > 0$ .
- (E2) The risk matrix on those  $n$  markets is

$$\Sigma_n(\rho) = \sigma^2((1 - \rho)I_n + \rho \mathbf{1}\mathbf{1}^\top), \quad \rho \in [0, 1].$$

- (E3) In the isolated regime each market receives free margin  $M$ , so isolated per-market depth is  $V = M/\sigma$ .
- (E4) Aggregate capital in the unified regime is the same total amount  $nM$ .
- (E5) Order flow is balanced across markets, so the venue carries a symmetric book  $q = x\mathbf{1}$ .

**Theorem 3.6** (Aggregate executable notional under equicorrelation). *Under Definition 3.5, the maximum total long-only notional  $L(n, \rho) := \max\{\mathbf{1}^\top q : q \in \mathbb{R}_{\geq 0}^n, M(q) \leq nM\}$  is*

$$L(n, \rho) = nV \Gamma_n(\rho), \quad \Gamma_n(\rho) := \sqrt{\frac{n}{1 + (n-1)\rho}}. \quad (5)$$

Equivalently, with diversification parameter  $\alpha := 1 - \rho \in [0, 1]$ ,

$$L(n, \alpha) = nV \sqrt{\frac{n}{n - (n-1)\alpha}}. \quad (6)$$

*Proof.* The constraint set  $\{q \in \mathbb{R}_{\geq 0}^n : q^\top \Sigma_n(\rho) q \leq (nM)^2\}$  is closed and convex. The objective  $\mathbf{1}^\top q$  is linear. Both the constraint set and the objective are invariant under the symmetric group  $S_n$  acting by permutation of coordinates. Hence there exists a maximizer in the fixed-point set of  $S_n$ , i.e. a maximizer of the form  $q = x\mathbf{1}$  with  $x \geq 0$ . (Rigorously: if  $q^*$

is any maximizer, its  $S_n$ -orbit-average  $\bar{q} := n!^{-1} \sum_{\pi \in S_n} \pi(q^*) = \bar{x}\mathbf{1}$  is feasible by convexity of the ball and achieves the same objective by linearity.)

For  $q = x\mathbf{1}$ ,

$$M(q)^2 = x^2 \mathbf{1}^\top \Sigma_n(\rho) \mathbf{1} = \sigma^2 x^2 n(1 + (n-1)\rho),$$

so the constraint  $M(q) \leq nM$  becomes  $\sigma x \sqrt{n(1 + (n-1)\rho)} \leq nM$ , which is binding at the maximum. Solving for  $x$ ,

$$x = \frac{nM}{\sigma \sqrt{n(1 + (n-1)\rho)}} = \frac{M}{\sigma} \sqrt{\frac{n}{1 + (n-1)\rho}} = V \Gamma_n(\rho).$$

The objective value is  $\mathbf{1}^\top q = nx = nV \Gamma_n(\rho)$ , giving (5). Equation (6) follows from  $\alpha = 1 - \rho$ .  $\square$

**Corollary 3.7** (Scaling bounds and the rejection of quadratic depth claims). *Under Definition 3.5,*

$$nV \leq L(n, \rho) \leq n^{3/2}V, \quad (7)$$

with:

- (a)  $L(n, \rho) = nV$  at  $\rho = 1$ ;
- (b)  $L(n, \rho) > nV$  for every  $\rho < 1$ ;
- (c)  $L(n, \rho) = n^{3/2}V$  at  $\rho = 0$ .

In particular, no theorem of this model produces a quadratic law in  $n$ . Expressions of the form  $nV + \alpha n(n-1)$  are modeling ansatzes unless separately derived; they are not implied by portfolio-margin algebra.

*Proof.* Since  $\rho \in [0, 1]$ ,  $1 \leq 1 + (n-1)\rho \leq n$ . Applying this to (5) gives (7). The endpoint statements follow immediately.  $\square$

**Remark 3.8** (Parameter discipline). The parameter  $\alpha = 1 - \rho$  in (6) is a diversification parameter in the interval  $[0, 1]$ . It is not an empirical constant and it is not identified by the theorem with any narrow range such as 0.05-0.15. If a calibration uses values in that range, that calibration is an empirical exercise layered on top of the theorem, not part of the theorem itself.

## 4 Cross-Margining Across Instrument Types

### 4.1 Offsets are real only under one risk matrix

**Proposition 4.1** (No double counting under matrix margin). *Let  $q \in \mathbb{R}^m$  and let  $\Sigma \succeq 0$ . Then the unified margin*

$$M(q)^2 = \sum_{i=1}^m \sigma_i^2 q_i^2 + 2 \sum_{1 \leq i < j \leq m} \Sigma_{ij} q_i q_j \quad (8)$$

*counts each offset exactly once. Within the quadratic model, any cross-margin schedule of the form*

$$M^{\text{rebate}}(q) = M^{\text{iso}}(q) - \sum_{(i,j) \in \mathcal{R}} D_{ij}(q_i, q_j), \quad M^{\text{iso}}(q) := \sqrt{\sum_{i=1}^m \sigma_i^2 q_i^2}, \quad (9)$$

*is safe only if it is algebraically equivalent to (8) for some positive-semidefinite matrix. Otherwise the same hedge can be credited multiple times.*

*Proof.* Equation (8) is the quadratic form generated by  $\Sigma$ . Each covariance term  $\Sigma_{ij}q_iq_j$  appears once in the double sum. There is no second channel through which the same pair can enter.

For (9), safety requires that the discounted margin remain a norm or seminorm generated by a positive-semidefinite quadratic form. If no such form exists, the rebate decomposition need not be subadditive and can assign credits whose sum exceeds the actual variance reduction. That is precisely double counting.  $\square$

**Remark 4.2** (Instrument-type interpretation). The proposition is operational for mixed books such as spot, perpetual, and event-linked positions. A long spot position hedged by a short perpetual position can receive an offset because the combined P&L variance falls. The same hedge cannot then receive a second independent credit merely because both positions are also related to an event market. All offsets must pass through one matrix or one scenario engine.

**Remark 4.3** (Relation to scenario-based margin). Proposition 4.1 is stated inside the quadratic model. Production portfolio-margin engines such as the SPAN methodology [9] replace the matrix by a finite set of joint-price scenarios and take the worst-case P&L across scenarios as the margin requirement; that construction is a coherent risk measure in the sense of Acerbi and Tasche [10] whenever the scenarios are closed under a convex-cone structure. The no-double-counting principle transfers: credits are real exactly when every offsetting pair affects the joint P&L inside one shared scenario generator, and any second credit from a schedule not derivable from that generator produces the same variance-budget inflation as the quadratic case.

## 4.2 Stress correlation and the Li-copula caveat

**Definition 4.4** (Stress matrix). Let  $\Sigma^{\text{base}}$  denote the central-body matrix used for routine margin and let  $\Sigma^{\text{stress}} \succeq \Sigma^{\text{base}}$  denote a stressed matrix used to cap tail credits. The operational limit for cross-margin depth is computed with  $\Sigma^{\text{stress}}$ .

**Remark 4.5** (Why a stress matrix is required). The Gaussian-copula register associated with Li [4] and the broader dependence critique of Embrechts, McNeil, and Straumann [3] point in the same direction for this paper's purpose: linear correlation and Gaussian dependence are central-body descriptions. They are not guarantees that offsets survive in the tail. When stressed tail dependence rises, the matrix in Section 3 must be replaced by a stress matrix or by a copula-derived upper envelope. The correct operational statement is therefore:

*cross-margining produces super-additive effective depth in normal regimes, and the bonus shrinks toward the additive baseline as tail correlation rises.*

**Corollary 4.6** (Crisis-correlation degradation). *In the equicorrelation model of Theorem 3.6, if average correlation rises from  $\rho$  to  $\rho^{\text{stress}} \in [\rho, 1]$ , then aggregate executable notional falls by the factor*

$$\frac{L(n, \rho^{\text{stress}})}{L(n, \rho)} = \sqrt{\frac{1 + (n-1)\rho}{1 + (n-1)\rho^{\text{stress}}}}. \quad (10)$$

As  $\rho^{\text{stress}} \rightarrow 1$ , the super-additive bonus disappears and  $L(n, \rho^{\text{stress}}) \rightarrow nV$ .

*Proof.* Substitute (5) at  $\rho$  and  $\rho^{\text{stress}}$  and simplify.  $\square$

**Remark 4.7** (What survives the tail regime). Corollary 4.6 is a useful sanity check. The tail regime does not create a negative-diversification theorem in this model. It erases the

bonus and returns the venue toward the additive baseline. Any further collapse requires a separate mechanism such as liquidity withdrawal, endogenous collateral reflexivity, or venue insolvency.

## 5 LP Incentives and the Fragmentation Trade-Off

### 5.1 Fees per unit displayed depth

**Definition 5.1** (Volume and fee multipliers). Let isolated market  $i$  generate baseline volume  $u_i$  and let total isolated volume be  $U = \sum_{i=1}^n u_i$ . Let  $f \in (0, 1)$  be the proportional fee rate. Aggregation can add cross-market routed flow or attract additional order flow; write the total volume multiplier as  $1 + \beta$  with  $\beta \geq 0$ , so aggregate venue volume is  $(1 + \beta)U$ . Let  $\Gamma_n(\rho)$  be the depth multiplier from (5).

**Proposition 5.2** (Fee flow per unit displayed depth). *In the symmetric equicorrelation model,*

$$y_{\text{iso}} = \frac{fU}{nV}, \quad y_{\text{agg}} = \frac{f(1 + \beta)U}{nV\Gamma_n(\rho)}. \quad (11)$$

Therefore aggregation increases fee flow per unit displayed depth if and only if

$$1 + \beta > \Gamma_n(\rho). \quad (12)$$

*Proof.* The isolated venue displays total depth  $nV$ . The aggregated venue displays total effective depth  $nV\Gamma_n(\rho)$  by Theorem 3.6. Dividing total fee flow by total displayed depth yields (11). Equation (12) is the resulting inequality.  $\square$

**Remark 5.3** (Tragedy-of-the-commons risk). Aggregation can deepen the venue faster than it grows fee flow. When (12) fails, fee flow per unit displayed depth falls even though total venue depth rises. This is the clean formulation of the tragedy-of-the-commons objection. It does not refute aggregation. It says that aggregation needs enough routed volume, enough fixed-cost savings, or enough risk-cost reduction to compensate for depth dilution.

### 5.2 Profitability on risk capital

**Definition 5.4** (LP profit comparison). Let  $c_0 > 0$  denote venue-level fixed operating cost,  $c_1 \geq 0$  per-market marginal cost, and let  $\Lambda_{\text{iso}}, \Lambda_{\text{agg}}$  denote non-fee risk costs (adverse selection, loss versus rebalancing, or other inventory drag) in the isolated and aggregated regimes. Define

$$\Pi_{\text{iso}} = fU - n(c_0 + c_1) - \Lambda_{\text{iso}}, \quad (13)$$

$$\Pi_{\text{agg}} = f(1 + \beta)U - (c_0 + nc_1) - \Lambda_{\text{agg}}. \quad (14)$$

**Theorem 5.5** (When aggregation is LP-positive). *Aggregation is LP-positive relative to fragmentation if and only if*

$$f\beta U + (n - 1)c_0 > \Lambda_{\text{agg}} - \Lambda_{\text{iso}}. \quad (15)$$

*Proof.* Subtract (13):

$$\Pi_{\text{agg}} - \Pi_{\text{iso}} = f\beta U + (n - 1)c_0 - (\Lambda_{\text{agg}} - \Lambda_{\text{iso}}).$$

The sign of this expression is positive exactly under (15).  $\square$

**Remark 5.6** (Depth dilution and capital return can disagree). Proposition 5.2 and Theorem 5.5 answer different questions. Equation (12) asks whether fee flow rises faster than displayed depth. Equation (15) asks whether total LP profit rises after fixed-cost amortization and risk-cost changes. Aggregation can fail the first test and pass the second.

## 6 Adversarial Stress Bounds

### 6.1 Concentrated withdrawals

**Definition 6.1** (Withdrawal gate). Let total effective depth before stress be  $D > 0$ . LP  $k$  controls share  $\omega_k$  of that depth, with  $\sum_k \omega_k = 1$ . Fix a stress window of fixed length. A gate parameter  $g \in [0, 1]$  caps the within-window withdrawal of each LP at fraction  $g$  of that LP's share. The proposition below bounds the depth loss within one such window; a sequence of windows can compound in the absence of new deposits, but that multi-period dynamic is outside the scope of this statement.

**Proposition 6.2** (Concentrated-withdrawal bound). *Let  $S$  be a coalition of withdrawing LPs. Under Definition 6.1, post-withdrawal depth satisfies*

$$D^+ \geq \left(1 - g \sum_{k \in S} \omega_k\right) D. \quad (16)$$

If each LP share is capped by  $\omega_{\max}$  and  $|S| = m$ , then

$$D^+ \geq (1 - gm\omega_{\max})D. \quad (17)$$

For first-order impact  $I(\Delta) = \Delta/D$ , slippage is inflated by at most the factor

$$\frac{I^+(\Delta)}{I(\Delta)} \leq \frac{1}{1 - gm\omega_{\max}}. \quad (18)$$

*Proof.* At most the fraction  $g\omega_k$  of depth attributable to LP  $k$  can leave in one window. Summing over  $S$  gives (16). Applying  $\omega_k \leq \omega_{\max}$  yields (17). Since first-order impact is inverse in depth, (18) follows.  $\square$

**Remark 6.3** (Sharpness and informative regime). The bound (17) is informative only for coalitions with  $gm\omega_{\max} < 1$ , i.e. for coalitions that in aggregate can pull less than the full depth; when  $gm\omega_{\max} \geq 1$  the bound degenerates to  $D^+ \geq 0$  and the governance implication is that the venue must either reduce  $\omega_{\max}$  or reduce  $g$  to keep any concentration-resilient operating regime. The slippage inflation (18) inherits this range. The bound controls first-order depth loss from an explicit withdrawal coalition. It does not control endogenous order-flow disappearance, collateral devaluation, or insolvency. Those channels belong to other models. For this paper's object, the result isolates the direct microstructure damage from concentration and gates it by observables  $(g, m, \omega_{\max})$ .

### 6.2 Fragmentation after stress

**Proposition 6.4** (Fragmentation cannot improve symmetric aggregate depth). *Fix  $\rho \in [0, 1]$  and isolated per-market depth  $V$ . Suppose a venue with  $n$  markets fragments into  $r$  disconnected components of sizes  $n_1, \dots, n_r$  with  $\sum_{j=1}^r n_j = n$ . If each component clears under the same equicorrelation parameter  $\rho$ , then*

$$\sum_{j=1}^r n_j V \Gamma_{n_j}(\rho) \leq n V \Gamma_n(\rho). \quad (19)$$

If  $\rho < 1$ , equality holds only for the trivial partition  $r = 1$ .

*Proof.* If  $\rho = 1$ , then  $\Gamma_n(1) = 1$  for every  $n$  and both sides of (19) equal  $nV$ . It remains to treat  $\rho < 1$ .

Define

$$\psi_\rho(x) := \Gamma_x(\rho) = \sqrt{\frac{x}{1 + (x-1)\rho}}$$

for real  $x \geq 1$ . Because

$$\frac{d}{dx} \left( \frac{x}{1 + (x-1)\rho} \right) = \frac{1 - \rho}{(1 + (x-1)\rho)^2} > 0,$$

$\psi_\rho$  is increasing for  $\rho < 1$ . Hence

$$nV\Gamma_n(\rho) = \sum_{j=1}^r n_j V\Gamma_n(\rho) \geq \sum_{j=1}^r n_j V\Gamma_{n_j}(\rho),$$

which is (19). □

**Remark 6.5** (Fragmentation response). Fragmentation is the mechanical opposite of within-venue aggregation. It removes cross-market netting and routed order flow. Under the symmetric model it weakly lowers aggregate executable notional. The inequality becomes strict whenever the venue had any genuine diversification benefit before fragmentation.

## 7 Corridor Networks Are Different

Within-venue aggregation is a shared-balance-sheet result. Corridor-network expansion is not. Each corridor is a negotiated edge with large fixed cost and long lead time. The governing question is which corridors are economically and politically worth building.

**Definition 7.1** (Cost-constrained corridor model). Let  $n$  denote the number of sovereign zones. Each corridor requires:

- (K1) expected negotiation cost  $\bar{c} = c_{\text{neg}}/p$ , where  $c_{\text{neg}}$  is direct corridor-development cost and  $p \in (0, 1)$  is formation success probability;
- (K2) lead time  $T_{\text{neg}}$  measured in years;
- (K3) net value filter: only corridors with expected surplus at least  $\bar{c}$  are built.

The order-of-magnitude regime motivating this model is

$$c_{\text{neg}} \sim 10^6 \text{ USD}, \quad T_{\text{neg}} \in [2, 4] \text{ years}, \quad p \in [0.7, 0.8],$$

broadly consistent with cost and duration estimates for bilateral investment treaties and preferential trade agreements documented in UNCTAD data [7] and in Dür, Baccini, and Elsig's preferential trade agreement design database [8]. Analogous cost-scarcity regimes produce the same  $O(n \log n)$  upper bound; the model depends on the inequality  $\bar{c} > 0$  rather than on specific numerical values.

**Proposition 7.2** (Sparse corridor scaling). *Assume:*

- (a) per-corridor expected cost  $\bar{c} > 0$  as in Definition 7.1;
- (b) candidate-corridor gross surplus values, for the entrant zone against the  $k$  incumbents, are evaluated by a rank-ordered Odlyzko-Tilly value model: after ordering candidates by expected bilateral surplus, the admissible set above the cost threshold  $\bar{c}$  has size at most  $\kappa \log k$  for a constant  $\kappa$  depending on the value curve and  $\bar{c}$ .

Then the number of economically admissible new corridors for the entrant at network size  $k$  is  $O(\log k)$ , and the total number of built corridors after  $n$  zones satisfies

$$E_n \leq \kappa \sum_{k=2}^n \log k = O(n \log n) \tag{20}$$

for a constant  $\kappa > 0$  depending on the surplus distribution and  $\bar{c}$ . If each built corridor contributes at most a uniformly bounded surplus, total realizable corridor value is also  $O(n \log n)$ .

*Proof.* Under assumption (b), for each fixed cost threshold  $\bar{c} > 0$  the number of candidate partners whose surplus with the entrant exceeds  $\bar{c}$  is  $O(\log k)$  as  $k \rightarrow \infty$  by construction of the rank-ordered value model. This is the corridor analogue of the Odlyzko-Tilly network-value argument [6]: value is concentrated in the top-ranked counterparties, so quadratic pair counting is not the correct baseline. The assumption is not the generic statement that IID heavy-tailed draws above a fixed threshold have logarithmic exceedance count; that statement would be false for ordinary Pareto sampling. Summing the per-entrant edge count over  $k = 2, \dots, n$  yields (20); the asymptotic  $\sum_{k=2}^n \log k = \Theta(n \log n)$  is Stirling. If each feasible corridor contributes at most a uniform upper-bound surplus, total realizable value has the same order.  $\square$

**Remark 7.3** (Relation to Odlyzko-Tilly). Proposition 7.2 is the corridor analogue of the weakly superlinear network-value argument of Odlyzko and Tilly [6]. It replaces the uniform pairwise valuation underlying Metcalfe scaling with a rank-ordered heterogeneous value model truncated by a cost filter. It is not a theorem that every corridor network must have exactly this value. It is a statement that once edge costs are large and pairwise value is heterogeneous with heavy-tailed rank statistics, quadratic pair counting is the wrong baseline.

**Remark 7.4** (Hub concentration and bounded land-grab). The cost parameters in Definition 7.1 create a natural hub-and-spoke bias. A complete mesh on  $n$  zones requires  $\binom{n}{2}$  bilateral negotiations. At order- $10^6$ -dollar per-edge cost and multi-year per-edge duration, that construction is not a realistic growth path. Sparse hub concentration follows from edge scarcity, and land-grab dynamics are bounded by the sovereign-negotiation bottleneck rather than by any software property of the corridor protocol.

**Remark 7.5** (Separation from within-venue aggregation). Nothing in Proposition 7.2 changes Theorem 3.6. The first theorem is about a venue's internal risk budget. The second proposition is about which inter-zone edges can exist at all. One cannot be used as a shortcut proof for the other.

## 8 Limitations

- (L1) **Quadratic margin model.** The exact formulas of Sections 3 and 4 are derived under a positive-semidefinite quadratic risk functional. Scenario engines, stressed expected shortfall, and path-dependent liquidation rules require a different analysis.
- (L2) **Central-body versus tail dependence.** The matrix model can overstate diversification in tail regimes. The paper therefore treats stress matrices and copula upper envelopes as mandatory for deploying cross-margin credits, not optional decorations.
- (L3) **Execution-layer simplification.** Displayed depth, routing latency, and market-impact curvature are suppressed into effective depth. That is deliberate. The paper isolates the balance-sheet contribution of aggregation instead of attempting a full exchange simulator.
- (L4) **LP cost aggregation.** Theorem 5.5 compresses adverse selection, loss versus rebalancing, and operational drag into  $\Lambda$ . The decomposition of  $\Lambda$  is venue specific.
- (L5) **Corridor model.** Proposition 7.2 is a cost-constrained network-formation statement. It is not a universal law of sovereignty. It says that once corridor creation is sparse and expensive,  $n^2$  is the wrong reference point.

## 9 Conclusion

Within-venue liquidity aggregation and inter-zone corridor growth are different objects. The first is governed by portfolio-risk geometry. The second is governed by sparse costly edge formation.

Within a single venue, cross-margining can produce super-additive effective depth, but only under stated conditions. The exact local condition is the Cauchy-Schwarz-strict inequality  $b_i(q) < \sigma_i R(q)$ , with the clean economic sufficient case  $b_i(q) < 0$  (incoming trade directly offsets the book). In the symmetric model the total-depth law is  $L(n, \rho) = nV \sqrt{n/(1 + (n-1)\rho)}$ : strictly super-additive for  $\rho < 1$ , additive at  $\rho = 1$ , and bounded above by  $n^{3/2}V$ . That is the correct replacement for unsupported quadratic depth claims.

For LPs, aggregation is beneficial only when routed flow and fixed-cost amortization outweigh additional risk cost. For risk, concentration and fragmentation admit explicit first-order bounds, and tail dependence must be handled with stressed matrices or copula caps.

Across sovereign zones, the correct comparison is not Metcalfe  $n^2$ . Once corridors require costly bilateral negotiation, weakly superlinear  $O(n \log n)$  scaling is the right order-of-growth benchmark.

## References

- [1] G. Angeris, A. Evans, T. Chitra, and S. Boyd. Optimal routing for constant function market makers. In *Proceedings of the ACM Conference on Economics and Computation (EC '22)*, 2022.
- [2] D. Duffie and H. Zhu. Does a central clearing counterparty reduce counterparty risk? *The Review of Asset Pricing Studies*, 1(1):74-95, 2011.
- [3] P. Embrechts, A. McNeil, and D. Straumann. Correlation and dependence in risk management: Properties and pitfalls. In *Risk Management: Value at Risk and Beyond*, pages 176-223. Cambridge University Press, 2002.
- [4] D. X. Li. On default correlation: A copula function approach. *The Journal of Fixed Income*, 9(4):43-54, 2000.
- [5] J. Millionis, C. C. Moallemi, T. Roughgarden, and A. L. Zhang. Automated market making and loss-versus-rebalancing. Working paper, 2022.
- [6] A. Odlyzko and B. Tilly. A refutation of Metcalfe's law and a better estimate for the value of networks and network interconnections. Preliminary manuscript, March 2005.
- [7] United Nations Conference on Trade and Development (UNCTAD). International Investment Agreements Navigator and World Investment Report: reference dataset for bilateral investment treaty count, duration, and termination. <https://investmentpolicy.unctad.org/international-investment-agreements>.
- [8] A. Dür, L. Baccini, and M. Elsig. The design of international trade agreements: Introducing a new dataset. *The Review of International Organizations*, 9(3):353-375, 2014.
- [9] Chicago Mercantile Exchange. Standard Portfolio Analysis of Risk (SPAN) methodology. Technical specification, Chicago Mercantile Exchange.
- [10] C. Acerbi and D. Tasche. On the coherence of expected shortfall. *Journal of Banking and Finance*, 26(7):1487-1503, 2002.